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LETTER TO THE EDITOR

Limit cycles as invariant functions of Lie groups

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Abstract. We establish necessary and sufficient group-theoretical criteria for the existence of limit cycles in differentiable flows defined by sets of N autonomous first-order ordinary differential equations. An example is given.

Useful necessary and sufficient conditions for the existence of limit cycles in oscillatory systems of more than two degrees of freedom have not proved easy to obtain. Recently, however, Steeb (1977a, b) has pointed out that if the set of autonomous differential equations defining a differentiable flow

$$\dot{x}^i - f^i(x) = 0, \qquad \dot{x}^i = \mathrm{d}x^i/\mathrm{d}t, \qquad -\infty < t < \infty,\ i = 1, \ldots, N \tag{1}$$

has a limit cycle solution, the limit cycle trajectory is an invariant function of a one-parameter group admitted by the equations. He has thereby stated in a potentially most useful form a necessary condition that limit cycles must satisfy. However as some closed trajectories in phase space $\{x\}$ are invariant functions but not limit cycles, and as some invariant functions in $\{x\}$ are not closed trajectories, a further characterization of limit cycles is needed. To this end we establish the following *Theorem*:

If T_0 is an invariant function of a one-parameter Lie group with generator

$$U_0 = \xi_0^i(x)\partial/\partial x^i \tag{2}$$

that is admitted by equations (1) that define a differentiable flow, then T_0 is a limit cycle trajectory if and only if

(i) U_0 is the generator of a compact group

and

(ii) The equations (1) admit no local one-parameter group of diffeomorphisms with generator

$$U_1 = \xi_1^i(x)\partial/\partial x^i \tag{3}$$

that is functionally independent of U_0 on T_0.

This result makes it possible to investigate limit cycles by using Lie's systematic methods for finding the generators of groups admitted by ordinary differential equations (Lie 1891), together with the theory of invariant solutions (Vessiot 1904, Ovsjannikov 1962, Bluman and Cole 1974). We now sketch a proof of the theorem.

Let a solution S_0 of (1) be defined by

$$g_0^i(x, t) = 0, \qquad i = 1, \ldots, N, \tag{4}$$

it being supposed that the functions g_0^i are differentiable. If there exists a set of solutions of (1) definable by

$$g^i(x, t;\ \delta a) \equiv g_0^i(x, t) + \delta a g_1^i(x, t) = 0, \qquad i = 1, \ldots, N, \tag{5}$$

where δa is an arbitrary infinitesimal parameter, then we may say that S_0 is a member of a local one-parameter family of solutions $S(\delta a)$. A local one-parameter family of trajectories $T(\delta a)$, perhaps corresponding to $S(\delta a)$, may be defined on the phase space $\{x\}$ by

$$h^i(x;\ \delta a) \equiv h_0^i(x) + \delta a h_1^i(x) = 0, \qquad i = 2, \ldots, N. \tag{6}$$

Now we may always write in (5)

$$g^i(x, t;\ \delta a) = (1 + \delta a W) g_0^i, \tag{7a}$$

$$W = \Omega^j(x, t)\partial/\partial x^j + \Upsilon(x, t)\partial/\partial t, \tag{7b}$$

and, in (6),

$$h^i(x, \delta a) = (1 + \delta a X) h_0^i, \tag{8a}$$

$$X = \chi^j(x)\partial/\partial x^j. \tag{8b}$$

Thus we may always suppose that the local one-parameter families (5), (6), are obtained by the action of a one-parameter family of first-order differential operators acting on one of their members. However, though the solution curves $S(\delta a)$ in $\{x, t\}$ are all diffeomorphic to the real line, the corresponding trajectories $T(\delta a)$ need not be diffeomorphic to each other. Let us then make the following *Definition*:

A differentiable trajectory T_0 is an *exceptional* trajectory if (1) admits no local one-parameter group of diffeomorphisms of phase space that converts T_0 into a one-parameter family of trajectories of (1).

If a trajectory T_0 is an exceptional one there are no differentiable neighbouring trajectories $T(\delta a)$ with the same topology. Because equations (1) determine a flow throughout phase space, if one establishes that T_0 is an exceptional trajectory, one immediately knows that T_0 is either a limit cycle or a 'limit line'. T_0 will be a limit cycle if and only if the one-parameter group generated by U_0 is compact. It will be a limit line if and only if the group generated by U_0 is non-compact.

Now equations (1) will admit a group of diffeomorphisms of phase space with generator

$$U = \xi^i(x)\partial/\partial x^i \tag{9a}$$

if and only if

$$[U, V] = 0, \tag{9b}$$

where

$$V = f^i(x)\partial/\partial x^i \tag{9c}$$

is the generator of the evolution group of (1). Suppose U_0 and U_1 satisfy equations (9). U_0 will have T_0 as an invariant function if and only if $\boldsymbol{\xi}_0 = (\xi_0^1, \ldots, \xi_0^N)$ is everywhere on T_0 tangent to T_0. U_1 will generate a one-parameter family of trajectories containing T_0 if and only if $\boldsymbol{\xi}_1 = (\xi_1^1, \ldots, \xi_1^N)$ is not tangent to T_0. Now $\boldsymbol{\xi}_1(x)$ is parallel to $\boldsymbol{\xi}_0(x)$ if and only if $\boldsymbol{\xi}_1(x) = \Phi(x)\boldsymbol{\xi}_0(x)$. Thus $\boldsymbol{\xi}_1$ is not tangent to T_0 if and only if on T_0 one has $\boldsymbol{\xi}_1(x) \neq \Phi(x)\boldsymbol{\xi}_0(x)$. This establishes the theorem.

As an example of the foregoing, consider the equations

$$\mathrm{d}\theta/\mathrm{d}t = 1, \qquad \mathrm{d}r/\mathrm{d}t = r - r^3 \tag{10}$$

where r and θ are polar coordinates. They define a real differentiable flow with generator

$$V = \partial/\partial\theta + (r - r^3)\partial/\partial r. \tag{11}$$

Solving the Lie determining equations arising from $[U, V] = 0$, one finds that, in their most general formal solution, ξ^θ is an arbitrary linear combination of the functions

$$r^\alpha (r^2 - 1)^{-\frac{1}{2}\alpha} \exp(-\alpha\theta)$$

and that ξ^r is an arbitrary linear combination of the functions

$$r^{\beta+1}(r^2 - 1)^{1-\frac{1}{2}\beta} \exp(-\beta\theta).$$

For the ξ^r, ξ^θ to define a real local diffeomorphism it is necessary that they be single-valued and real so one must choose the functions with $\alpha = \beta = 0$. One then obtains as the most general allowed generator

$$U = a\partial/\partial\theta + br(r^2 - 1)\partial/\partial r, \qquad a, b \text{ constants.} \tag{12}$$

An invariant curve $h(r, \theta)$ of a group with generator U satisfies $Uh| = 0$. Here this has the obvious solution $r = 1$. On it the only non-vanishing U is $U_0 = \partial/\partial\theta$, so this trajectory is an exceptional one. Because θ is a cyclic coordinate, the trajectory is obviously a limit cycle.

The remaining invariant curves are obtained when $U_0 = \partial/\partial\theta + (r - r^3)\partial/\partial r$, that is when $U_0 = V$, and may be given in the form

$$r^2/r^2 - 1 = \exp 2(\theta + c), \qquad r \neq 1. \tag{13}$$

Any U, (12), with $a \neq b$ is functionally independent of $U_0 = V$, so the trajectories given by (13) are not exceptional ones.

In closing we would emphasise that it has proved possible to use an investigation of the Lie *algebra* admitted by a differentiable flow to determine whether the flow does or does not contain exceptional trajectories. This is because a study of infinitesimal rather than finite transformations admitted by the flow has sufficed to make the distinction between neighbouring trajectories which are or are not diffeomorphic. It is not yet evident whether one can also distinguish between cyclic and acyclic exceptional trajectories by investigations of the Lie algebra of transformations in $\{x, t\}$ admitted by the governing differential equations, as has proved possible in the case of the harmonic oscillator (Wulfman and Wybourne, 1976).

References

Bluman G W and Cole J P 1974 *Similarity Methods for Differential Equations* (NY: Springer)

Lie S 1891 *Differentialgleichungen* Leipzig (1967 reprint, NY: Chelsea)

Ovsjannikov L V 1962 *Gruppovye svoystva differentsialny uravneni* (Novosibirsk) (1969 *Group Properties of Differential Equations* transl. G W Bluman Vancouver: University of British Columbia)

Steeb W-H 1977a *J. Phys. A: Math. Gen.* **10** L221

—— 1977b *Lett. Math. Phys.* **2** 171

Vessiot E 1904 *Acta Math.* **28** 307

Wulfman C E and Wybourne B G 1976 *J. Phys. A: Math. Gen.* **9** 507